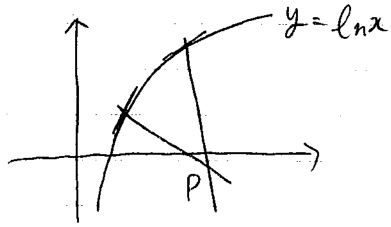
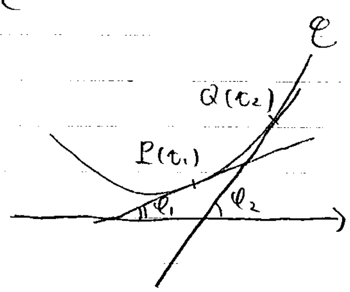
微分幾何 (D.G.)
Vector G

Lct 1. 曲率 / 曲率円
2. 閉曲線の面積



Lecture 1

$$\mathcal{C} \begin{cases} x = x(t) \\ y = y(t) \end{cases}$$



$$\frac{\Delta\varphi}{\Delta s} \xrightarrow[\Delta \to 0]{} K_s$$

$$\frac{d\varphi}{dt}$$

$\tan\varphi = \dfrac{\dot{y}}{\dot{x}}$ より $\arctan\dfrac{\dot{y}}{\dot{x}} = \varphi$

~~$\eta = \tan\theta$ とし~~ $\eta' = \dfrac{1}{\cos^2\theta}$ ~~$= 1 + \tan^2\theta$~~

~~$\therefore \dfrac{d\eta}{d\theta} = 1 + \tan^2\theta$ より~~ $\dfrac{d\theta}{d\eta} = \dfrac{1}{1+\eta^2}$

$\therefore \varphi = \arctan\dfrac{\dot{y}}{\dot{x}}$ より

$$\frac{d\varphi}{dt} = \frac{1}{1+\left(\frac{\dot{y}}{\dot{x}}\right)^2} \cdot \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2}$$

$$= \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}$$

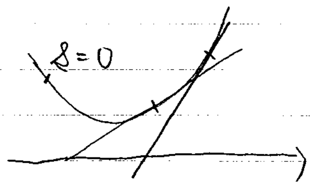
弧長 $s = \int_a^{t_0} \sqrt{\dot{x}^2+\dot{y}^2}\,dt$ より

$$\frac{ds}{dt_0} = \sqrt{\dot{x}^2+\dot{y}^2}$$

$$\therefore \frac{d\varphi}{ds} = \frac{d\varphi}{dt}\cdot\frac{dt}{ds}$$

$$= \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2+\dot{y}^2}\cdot\frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}} = \frac{\begin{vmatrix}\dot{x} & \ddot{x}\\ \dot{y} & \ddot{y}\end{vmatrix}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}$$

$$= \frac{|\dot{p}\ \ddot{p}|}{\|\dot{p}\|^3}$$

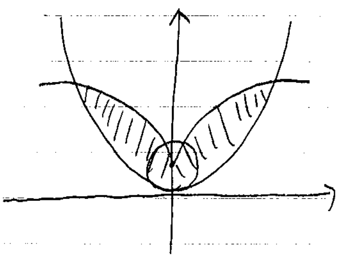
[

弧長をパラメータに． $\Rightarrow \frac{dt}{ds} = 1$]

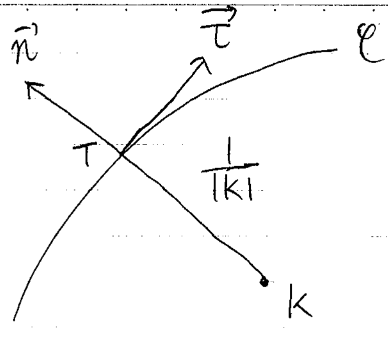
$$\therefore \text{曲率 } K(t) = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}$$

曲率Kについて

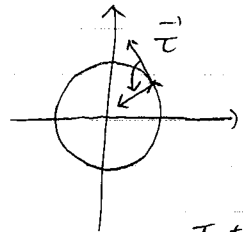
$\frac{1}{|K|}$ を曲率半径．



※（入試の）実用例



$\mathcal{C}$ の向き付け



$\vec{\tau}$ を正規化して $\vec{u} = \dfrac{\vec{\tau}}{|\vec{\tau}|}$

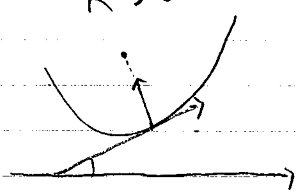
$\vec{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{u}$

正規直交基底

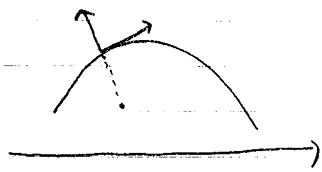
$\underbrace{\text{artho}}_{\text{直交}}\underbrace{\text{normal}}_{\text{正規}}$ basis

$$\overrightarrow{OK} = \overrightarrow{OT} + \overrightarrow{TK}$$
$$= \begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{|K|}\left(\pm \frac{\vec{n}}{|\vec{n}|} \right)$$

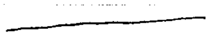
$K > 0$



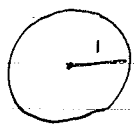
$K < 0$



ex.



$K = 0$



$K = 1$

$$K = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}$$

$x = \cos t,\ y = \sin t$ ← 弧長パラメータ表示!

$\dot{x} = -\sin t \quad \dot{y} = \cos t$

$\ddot{x} = -\cos t \quad \ddot{y} = -\sin t$

$\therefore -K = -\sin t(-\sin t) - (-\cos t)\cdot \cos t = 1$

$\therefore K = 1$, 曲率半径 $\rho = \dfrac{1}{K} = 1$

Task 1.1 (2) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ astroid

× asteroid (小惑星帯)

$x^{\frac{1}{3}} = \cos\theta,\quad y^{\frac{1}{3}} = \sin\theta$

$x = \cos^3\theta,\quad y = \sin^3\theta$

$\dot{x} = 3\cos^2\theta(-\sin\theta) \qquad \dot{y} = 3\sin^2\theta\cos\theta$

$\left(\therefore \dfrac{\dot{y}}{\dot{x}} = -\tan\theta\right)$

$\ddot{x} = -3(-2\cos\theta\sin^2\theta + \cos^2\theta\cos\theta)$

$= -3\cos\theta(\cos^2\theta - 2\sin^2\theta)$

$\ddot{y} = 3(2\sin\theta\cos^2\theta + \sin^2\theta(-\sin\theta))$

$= 3\sin\theta(2\cos^2\theta - \sin^2\theta)$

$\therefore \begin{vmatrix} \dot{x} & \ddot{x} \\ \dot{y} & \ddot{y} \end{vmatrix} = -3\sin\theta\cos^2\theta\cdot 3\sin\theta(2\cos^2\theta - \sin^2\theta)$

$+ 3\cos\theta(\cos^2\theta - 2\sin^2\theta)\cdot 3\sin^2\theta\cos\theta$

$= 9\sin^2\theta\cos^2\theta(-2\cos^2\theta + \sin^2\theta + \cos^2\theta - 2\sin^2\theta)$

$= -9\sin^2\theta\cos^2\theta$

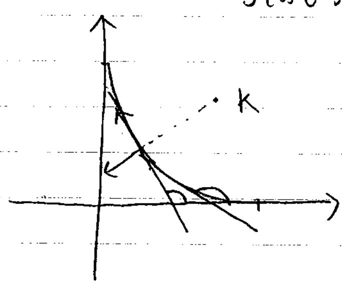
$\dot{x}^2 + \dot{y}^2 = 9\cos^2\theta\sin^2\theta\cdot 1$ より

$(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}} = 27\cos^3\theta\sin^3\theta$

$\therefore K = \dfrac{9\sin^2\theta\cos^2\theta}{27\cos^3\theta\sin^3\theta}$

$= -\dfrac{1}{3\cos\theta\sin\theta} \qquad \rho = 3\cos\theta\sin\theta$

(Tを第1象限でとる)



$K < 0.$

$\vec{\tau} = 3\sin\theta\cos\theta\begin{pmatrix} -\cos\theta \\ \sin\theta \end{pmatrix}$

$\vec{\tau}' = \begin{pmatrix} -\cos\theta \\ \sin\theta \end{pmatrix}$

$\vec{n} = \begin{pmatrix} -\sin\theta \\ -\cos\theta \end{pmatrix}$

曲率中心 K(ξ, η) とすると

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos^3\theta \\ \sin^3\theta \end{pmatrix} + 3\cos\theta\sin\theta \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

$$\xi = \cos^3\theta + 3\sin^2\theta\cos\theta = \cos\theta(\cos^2\theta + 3\sin^2\theta)$$
$$\eta = \sin^3\theta + 3\sin\theta\cos^2\theta = \sin\theta(\sin^2\theta + 3\cos^2\theta)$$

cf. $\begin{cases} \xi + \eta = (\cos\theta + \sin\theta)^3 \\ \xi - \eta = (\cos\theta - \sin\theta)^3 \end{cases}$

Kの中心をξとして、ξを$\frac{\pi}{4}$回転

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \xi - \eta \\ \xi + \eta \end{pmatrix}$$

$\frac{1}{\sqrt{2}}(\xi - \eta) = \bar{x}$, $\frac{1}{\sqrt{2}}(\xi + \eta) = \bar{y}$ とすれば

$\xi - \eta = \sqrt{2}\,\bar{x}$, $\xi + \eta = \sqrt{2}\,\bar{y}$

$$\therefore (\xi + \eta)^{\frac{2}{3}} + (\xi - \eta)^{\frac{2}{3}}$$
$$= (\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2 = 2$$

$\therefore$

$$(\sqrt{2}\,\bar{x})^{\frac{2}{3}} + (\sqrt{2}\,\bar{y})^{\frac{2}{3}} = 2$$
$$2^{\frac{1}{3}}\bar{x}^{\frac{2}{3}} + 2^{\frac{1}{3}}\bar{y}^{\frac{2}{3}} = 2$$
$$\bar{x}^{\frac{2}{3}} + \bar{y}^{\frac{2}{3}} = 2^{\frac{2}{3}}$$

cycloid

$$\mathcal{C}\ \begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$$

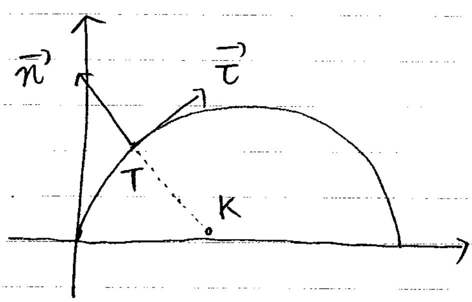
$$\dot{x} = a(1-\cos t) \qquad \dot{y} = a\sin t$$
$$\ddot{x} = a\sin t \qquad \ddot{y} = a\cos t$$

$$\begin{aligned} \dot{x}^2 + \dot{y}^2 &= a^2(1-\cos t)^2 + a^2\sin^2 t \\ &= a^2(2 - 2\cos t) \\ &= 2a^2(1-\cos t) = 2a^2 \cdot 2\sin^2\tfrac{t}{2} \end{aligned}$$

$$\begin{aligned} \dot{x}\ddot{y} - \ddot{x}\dot{y} &= a(1-\cos t)a\cos t - a\sin t\, a\sin t \\ &= a^2(\cos t - \cos^2 t - \sin^2 t) \\ &= a^2(\cos t - 1) \\ &= a^2\left(-2\sin^2\tfrac{t}{2}\right) \\ &= -2a^2\sin^2\tfrac{t}{2} \end{aligned}$$

$$(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}} = 2^3 a^3 \sin^3\tfrac{t}{2}$$

$$\therefore\ \kappa = \frac{-2a^2\sin^2\frac{t}{2}}{2^3a^3\sin^3\frac{t}{2}} = -\frac{1}{4a\sin\frac{t}{2}}$$



$\rho = 4a\left|\sin\frac{t}{2}\right|$

$\vec{\tau} = \begin{pmatrix} 1-\cos t \\ \sin t \end{pmatrix}$

法線 vector は

$\begin{pmatrix} \sin t \\ \cos t - 1 \end{pmatrix}$ (y成分<0) とできる

これを $\vec{m}$ として

$|\vec{m}|^2 = 2 - 2\cos t = 4\sin^2\frac{t}{2}$

$$\begin{aligned} \vec{OK} = \vec{OT} + \vec{TK} &= a\begin{pmatrix} t-\sin t \\ 1-\cos t \end{pmatrix} + 4a\left|\sin\tfrac{t}{2}\right| \frac{1}{2\left|\sin\frac{t}{2}\right|}\begin{pmatrix} \sin t \\ \cos t - 1 \end{pmatrix} \\ &= a\begin{pmatrix} t-\sin t \\ 1-\cos t \end{pmatrix} + 2a\begin{pmatrix} \sin t \\ \cos t - 1 \end{pmatrix} = a\begin{pmatrix} t+\sin t \\ -1+\cos t \end{pmatrix} \end{aligned}$$

曲率中心Kの軌跡

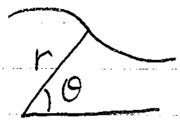
$\mathcal{C}\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases}$ と $\mathcal{E}\begin{cases} \xi = t + \sin t \\ \eta = -1 + \cos t \end{cases}$ について

$$\xi = t + \sin t$$
$$= (t + \pi) - \sin(t + \pi)$$
$$\eta = 1 - \cos(t + \pi) - 2$$

$\xi + \pi = \bar{x}$, $\eta + 2 = \bar{y}$ とすれば
$t + \pi = s$ にて,
$\bar{x} = s - \sin s$, $\bar{y} = 1 - \cos s$
これは $\mathcal{C}$ を平行移動した cycloid

Bernouli
$\begin{cases} \text{等時ふりこ} \\ \text{最速降下線} \end{cases}$

Task 1.2 $r = f(\theta)$



(1) ℓ を p-表示すれば

$x = r\cos\theta,\ y = r\sin\theta$

$\dot{x} = \dot{r}\cos\theta - r\sin\theta,\quad \dot{y} = \dot{r}\sin\theta + r\cos\theta$

$\ddot{x} = \ddot{r}\cos\theta - \dot{r}\sin\theta - \dot{r}\sin\theta - r\cos\theta$

$= \ddot{r}\cos\theta - 2\dot{r}\sin\theta - r\cos\theta$

$\ddot{y} = \ddot{r}\sin\theta + \dot{r}\cos\theta + \dot{r}\cos\theta - r\sin\theta$

$= \ddot{r}\sin\theta + 2\dot{r}\cos\theta - r\sin\theta$

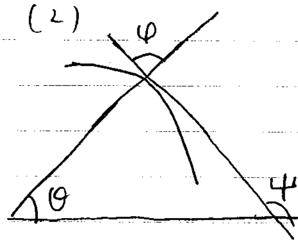
$\dot{x}\ddot{y} - \ddot{x}\dot{y} = (\dot{r}\cos\theta - r\sin\theta)(\ddot{r}\sin\theta + 2\dot{r}\cos\theta - r\sin\theta)$
$\quad - (\ddot{r}\cos\theta - 2\dot{r}\sin\theta - r\cos\theta)(\dot{r}\sin\theta + r\cos\theta)$
$= -\ddot{r}r + 2\dot{r}^2 + r^2$

$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2$

$\therefore K = \dfrac{-\ddot{r}r + 2\dot{r}^2 + r^2}{(\dot{r}^2 + r^2)^{\frac{3}{2}}}$ より

$\rho = \dfrac{(\dot{r}^2 + r^2)^{\frac{3}{2}}}{|r^2 + 2\dot{r} - 2r\ddot{r}|}$

(2)



$P[r, \theta]$ として $x = r\cos\theta,\ y = r\sin\theta$

$\therefore \tan\theta = \dfrac{y}{x}$

$\begin{cases} \dot{x} = \dot{r}\cos\theta - r\sin\theta \\ \dot{y} = \dot{r}\sin\theta + r\cos\theta \end{cases}$ より

$\tan\psi = \dfrac{\dot{r}\sin\theta + r\cos\theta}{\dot{r}\cos\theta - r\sin\theta}$

$= \dfrac{\dot{r}\tan\theta + r}{\dot{r} - r\tan\theta}$

$\psi = \theta + \varphi$ より $\varphi = \psi - \theta$

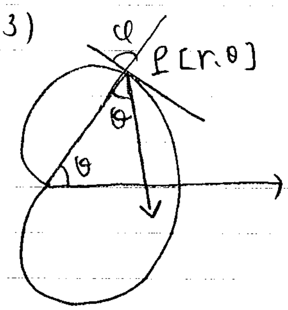
$\tan\varphi = \tan(\psi - \theta)$

$= \dfrac{\tan\psi - \tan\theta}{1 + \tan\psi\tan\theta}$

$= \dfrac{\dfrac{\dot{r}\sin\theta + r\cos\theta}{\dot{r}\cos\theta - r\sin\theta} - \tan\theta}{1 + \dfrac{\dot{r}\tan\theta + r}{\dot{r} - r\tan\theta}\cdot\tan\theta}$

$$\tan\varphi = \frac{\dot{r}\tan\theta + r - \tan\theta(\dot{r} - r\tan\theta)}{\dot{r} - r\tan\theta + \tan\theta(\dot{r}\tan\theta + r)}$$
$$= \frac{r + r\tan^2\theta}{\dot{r} + \dot{r}\tan^2\theta} = \frac{r}{\dot{r}}$$

(3)



$\dot{r} = -a\sin\theta$ より P$[r, \theta]$ における
動径 OP と接線とのなす角を φ

$$\tan\varphi = \frac{r}{\dot{r}}$$
$$= \frac{a(1+\cos\theta)}{-a\sin\theta}$$
$$= -\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$
$$= -\frac{1}{\tan\frac{\theta}{2}} \quad \left(= -\cot\frac{\theta}{2}\right)$$
$$= \tan\left(\frac{\theta}{2} + \frac{\pi}{2}\right) \quad \therefore\ \varphi = \frac{\theta}{2} + \frac{\pi}{2}$$

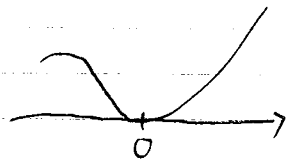
入射角, 反射角はいずれも $\pi - \varphi$ だから
それぞれのなす角は
$$\pi - 2(\pi - \varphi) = 2\varphi - \pi = \theta$$

Task 1.3 fが2回連続的可微分だから

f, f' は dif'ble であり、 f, f', f'' は contin

C : y=f(x) の曲率 K は

$$K = \frac{f''}{(1+f'^2)^{\frac{3}{2}}} \quad \text{だから} \quad \rho = \frac{(1+f'^2)^{\frac{3}{2}}}{|f''|}$$



Cは O で x軸に接するから

$f(0) = f'(0) = 0$

よって、Oにおける曲率半径 $\rho_{x=0} = \dfrac{1}{|f''(0)|}$

ここで、f についての仮定を

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(\theta x)}{2!}x^2 \quad \exists\, \theta \in (0,1)$$

$$= \frac{1}{2} f''(\theta x)x^2 \quad \therefore f''(\theta x) = \frac{2y}{x^2}$$

f'' は contin だから

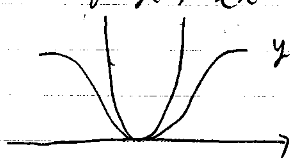
$$f''(0) = \lim_{x\to 0} f''(\theta x) = \lim_{x \to 0} \frac{2y}{x^2}$$

$$\therefore \rho_{x=0} = \frac{1}{|f''(0)|} = \lim_{x\to 0} \left| \frac{x^2}{2y} \right|$$

$1 + kx^2 \leq \cos x$

$\Leftrightarrow 1 - \cos x \leq -kx^2$ $-k = l$ とおく

$y = g(x) = lx^2$



$y = f(x) = 1 - \cos x$

$f(x) = 1 - \cos x$ $g(x) = lx^2$ は

いずれも O で x軸に接するから

$$\rho^f_{x=0} = \lim_{x\to 0}\left|\frac{x^2}{2y}\right| = \lim_{x \to 0} \frac{x^2}{2(1-\cos x)}$$

$$= 1$$

また

$$\rho^g_{x=0} = \lim_{x\to 0} \left|\frac{x^2}{2y}\right| = \frac{1}{2l}$$

$\rho^g_{x=0} \leq \rho^f_{x=0}$ より

$$\frac{1}{2l} \leq 1 \quad \therefore l \geq \frac{1}{2}$$

$$\therefore k \leq -\frac{1}{2}$$

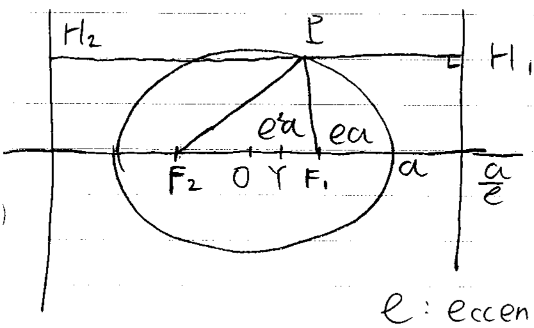
Task 1.5 $\quad \mathcal{E}: \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$x = a\cos t,\quad y = b\sin t$$
$$\dot{x} = -a\sin t,\quad \dot{y} = b\cos t$$
$$\ddot{x} = -a\cos t,\quad \ddot{y} = -b\sin t$$

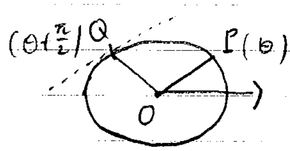
$$\dot{x}^2 + \dot{y}^2 = a^2\sin^2 t + b^2\cos^2 t$$
$$\dot{x}\ddot{y} - \ddot{x}\dot{y} = ab$$
$$\kappa = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}}$$
$$= \frac{a^4 b^4}{(a^4 y^2 + b^4 x^2)^{\frac{3}{2}}}$$

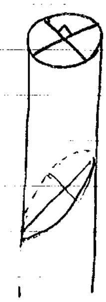
よって

$$\rho = \frac{(a^4 y^2 + b^4 x^2)^{\frac{3}{2}}}{a^4 b^4}$$
$$\left(= \frac{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}}{ab} \right)$$



[共役半径定理]
$PF_1 \cdot PF_2 = OQ^2$





e: eccentricity

$r_1 \cdot r_2 = OQ^2$ を示す

$P(\theta)$ とすれば $Q(\theta + \frac{\pi}{2})$

$$a\cos\left(\theta + \frac{\pi}{2}\right) = -a\sin\theta$$

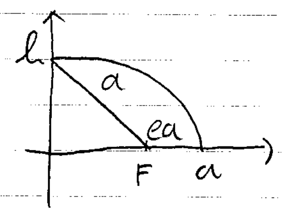
$$b\sin\left(\theta + \frac{\pi}{2}\right) = b\cos\theta$$

$$\therefore OQ^2 = a^2\sin^2\theta + b^2\cos^2\theta$$

$$PF_1 = ePH_1 = e\left(\frac{a}{e} - x\right)$$

$$PF_2 = ePH_2 = e\left(x + \frac{a}{e}\right)$$

$$\begin{aligned} PF_1 \cdot PF_2 &= (a - ex)(a + ex) \\ &= a^2 - e^2x^2 \\ &= a^2 - e^2a^2\cos^2\theta \end{aligned}$$



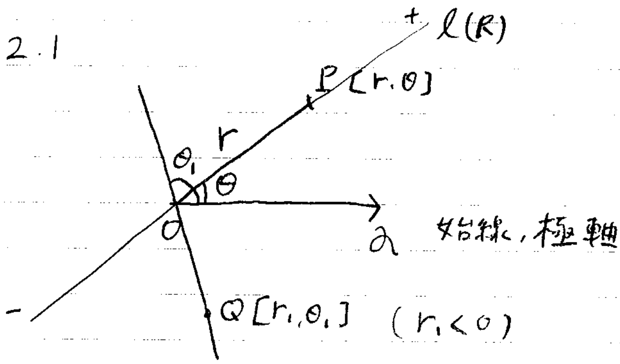
$e^2a^2 = a^2 - b^2$ より

$$\begin{aligned} PF_1 \cdot PF_2 &= a^2 - (a^2 - b^2)\cos^2\theta \\ &= a^2(1 - \cos^2\theta) + b^2\cos^2\theta \\ &= a^2\sin^2\theta + b^2\cos^2\theta \end{aligned}$$

$$\rho = \frac{(r_1 r_2)^{\frac{3}{2}}}{ab} = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{\frac{3}{2}}}{ab}$$

Lecture 2. Area

Task 2.1



(1) $r = 3 + \cos\theta$ ⋯ C

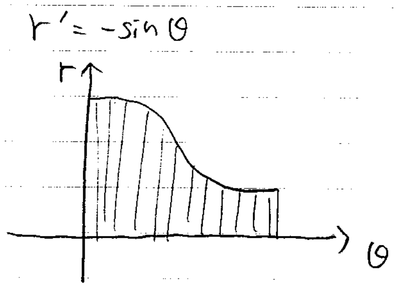
$f(\theta) = 3 + \cos\theta$ とすれば

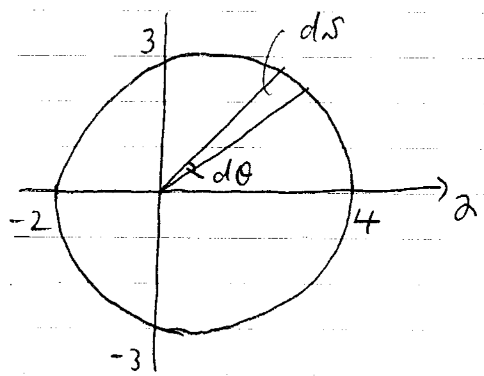
$f(-\theta) = 3 + \cos(-\theta) = 3 + \cos\theta$

よって，C は極軸対称だから

$0 \leq \theta \leq \pi$ で考えれば十分

$r' = -\sin\theta$





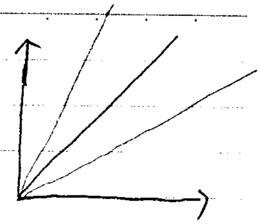
$dS = \frac{1}{2} r^2 d\theta$ より

$$S = 2\int_{\theta=0}^{\theta=\pi} dS$$

$$= \int_0^{\pi} (3+\cos\theta)^2 d\theta$$

$$= \int_0^{\pi} \left(9 + 6\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right) d\theta$$

$$= \frac{19}{2}\pi$$



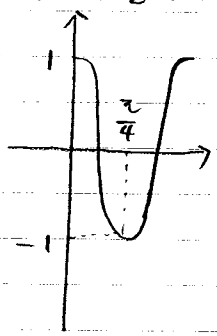
(2) $r = \cos 4\theta$

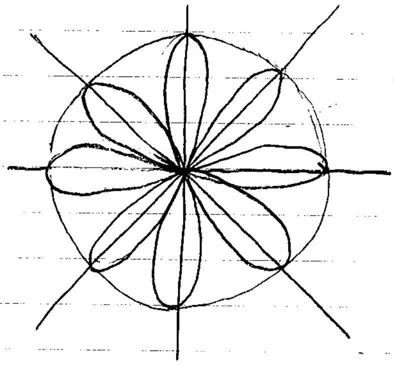
$g(\theta) = \cos 4\theta$ とすると

$$g\left(\theta + \frac{\pi}{4}\right) = \cos(4\theta + \pi) = -\cos 4\theta = -g(\theta)$$

$$g\left(\frac{\pi}{2} - \theta\right) = \cos(2\pi - 4\theta) = \cos 4\theta = g(\theta)$$

周期$\frac{\pi}{2}$をもち、$\theta = \frac{\pi}{4}$に関して対称





$r' = -4\sin 4\theta$

$$S = 16\int_{\theta=0}^{\theta=\frac{\pi}{8}} dS$$

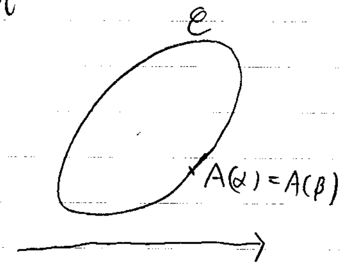
$$= 8\int_0^{\frac{\pi}{8}} \frac{1}{2}(1+\cos 8\theta)\,d\theta$$

$$= 4\int_0^{\frac{\pi}{8}} d\theta = \frac{\pi}{2}$$

閉曲線の面積の基本公式

$$\mathscr{C} : \begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

$t \in [\alpha, \beta]$

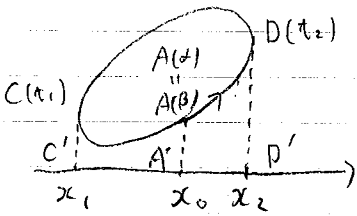


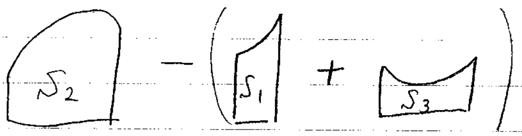
$dy = \underline{f'(x)}\, dx$
微分係数

$$S_{\mathscr{C}} = \frac{1}{2}\int_\alpha^\beta (x\,dy - y\,dx) = \frac{1}{2}\int_\alpha^\beta \begin{vmatrix} x & dx \\ y & dy \end{vmatrix}$$

$$= \frac{1}{2}\int_\alpha^\beta \left(x\frac{dy}{dt} - y\frac{dx}{dt} \right) dt = \frac{1}{2}\int_\alpha^\beta \begin{vmatrix} x & \dot{x} \\ y & \dot{y} \end{vmatrix} dt$$

また, $$S_{\mathscr{C}} = \frac{1}{2}\int_\alpha^\beta x^2 \frac{d}{dx}\left(\frac{y}{x}\right) dt$$





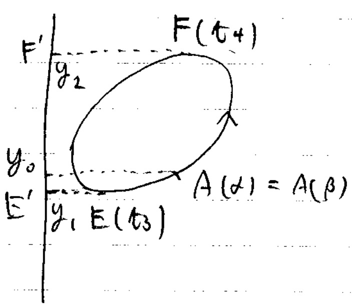
$$S_1 = S_{AA'D'D} = \int_{x_0}^{x_2} y\,dx = \int_\alpha^{t_2} y\frac{dx}{dt}\,dt$$

$$S_2 = S_{CC'D'D} = \int_{x_1}^{x_2} y\,dx = -\int_{x_2}^{x_1} y\,dx = -\int_{t_2}^{t_1} y\frac{dx}{dt}\,dt$$

$$S_3 = S_{CC'A'A} = \int_{x_1}^{x_0} y\,dx = \int_{t_1}^{\beta} y\frac{dx}{dt}\,dt$$

$$\therefore S = S_2 - S_1 - S_3 = -\int_{t_1}^{t_2} - \int_\alpha^{t_2} - \int_{t_1}^\beta$$

$$= -\left(\int_\alpha^{t_1} + \int_{t_1}^{t_2} + \int_{t_2}^\beta \right) = -\int_\alpha^\beta$$



また, $$S = \int_{y_0}^{y_2} + \int_{y_1}^{y_0} - \int_{y_1}^{y_2}$$

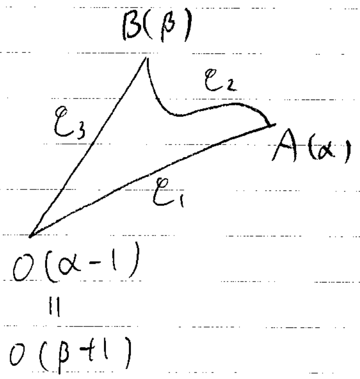
$$= \int_\alpha^{t_4} + \int_{t_3}^\beta - \left(-\int_{t_4}^{t_3} \right)$$

$$= \int_\alpha^\beta$$

$$\therefore S = \frac{1}{2}\left\{ \int_\alpha^\beta x \frac{dy}{dt} dt - \int_\alpha^\beta y \frac{dx}{dt} dt \right\}$$
$$= \frac{1}{2}\int_\alpha^\beta (x\dot{y} - \dot{x}y) dt$$
$$= \frac{1}{2}\int_\alpha^\beta (x dy - y dx)$$

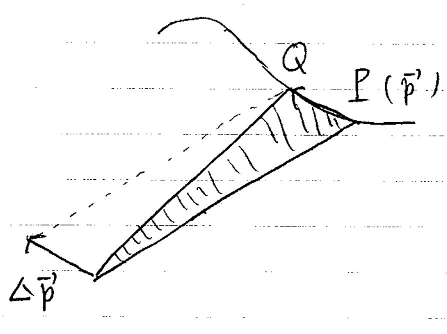
$x\dot{y} - \dot{x}y$ について

$$\frac{d}{dt}\left(\frac{y}{x}\right) = \frac{x\dot{y} - \dot{x}y}{x^2} \quad \text{より}$$
$$S = \frac{1}{2}\int_\alpha^\beta x^2 \frac{d}{dx}\left(\frac{y}{x}\right) dt$$



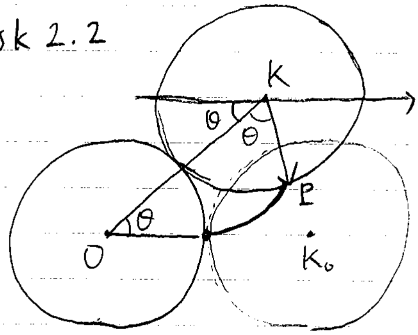
$$S = \frac{1}{2}\left(\int_{\ell_1} + \int_{\ell_2} + \int_{\ell_3} \right)$$
$$\int_{\ell_1} = \int_{\alpha-1}^{\alpha} x^2 \frac{d}{dx}\left(\frac{y}{x}\right) dt = 0. \quad \text{同様に} \int_{\ell_3} = 0.$$
$$\therefore S = \frac{1}{2}\int_\alpha^\beta (x dy - y dx)$$

cf.



$$\vec{p} \times \Delta\vec{p} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \times \begin{pmatrix} \Delta x \\ \Delta y \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 0 \\ x\Delta y - y\Delta x \end{pmatrix}$$
$$\therefore \triangle OPQ = \frac{1}{2}(x\Delta y - y\Delta x)$$

Task 2.2



$$\overrightarrow{OP} = \overrightarrow{OK} + \overrightarrow{KP}$$

$$\overrightarrow{OK} = 2\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$$

$\arg \overrightarrow{KP} = \pi + 2\theta$ より

$$\overrightarrow{KP} = \begin{pmatrix}-\cos 2\theta\\ -\sin 2\theta\end{pmatrix}$$

$\therefore$ P(x,y) として $\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}2\cos\theta - \cos 2\theta\\ 2\sin\theta - \sin 2\theta\end{pmatrix}$

$$x = 2\cos\theta - (2\cos^2\theta - 1)$$

$$\therefore x - 1 = 2\cos\theta(1-\cos\theta)$$

$$y = 2\sin\theta(1-\cos\theta)$$

$$\therefore \begin{pmatrix}x-1\\ y\end{pmatrix} = (1-\cos\theta)\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$$

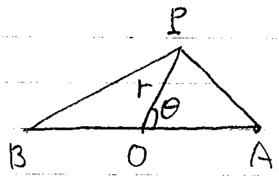
$$r = 1-\cos\theta.$$

よって、点$(1,0)$を極とすれば

極方程式 $r = 2(1-\cos\theta)$

$\underline{S = 6\pi.}$

Task 2.3



極平面
Cosine Rule
(余弦定理)

$PA \cdot PB = a^2$

$P[r, \theta]$ とする

$PA^2 = r^2 + a^2 - 2ar\cos\theta$

$PB^2 = r^2 + a^2 + 2ar\cos\theta$

$\therefore PA^2 \cdot PB^2 = (r^2+a^2)^2 - 4a^2r^2\cos^2\theta = a^4$

$r^4 + 2a^2r^2 - 4a^2r^2\cos^2\theta = 0$

$r \neq 0$ なので

$r^2 = 4a^2\cos^2\theta - 2a^2$

$= 2a^2(2\cos^2\theta - 1)$

$= 2a^2\cos 2\theta$

$-\pi \leq \theta \leq \pi \iff -2\pi \leq 2\theta \leq 2\pi$ において

$\cos 2\theta \geq 0 \iff -2\pi \leq 2\theta \leq -\frac{3}{2}\pi,$

$-\frac{\pi}{2} \leq 2\theta \leq \frac{\pi}{2},$

$\frac{3}{2}\pi \leq 2\theta \leq 2\pi$

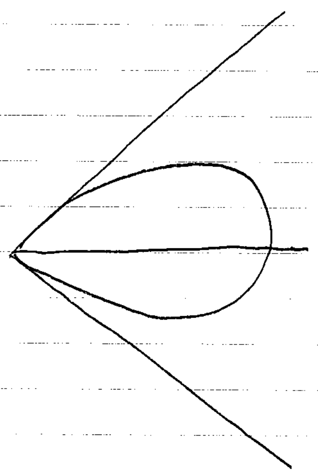
$\therefore -\pi \leq \theta \leq -\frac{3}{4}\pi, \ -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}, \ \frac{3}{4}\pi \leq \theta \leq \pi$

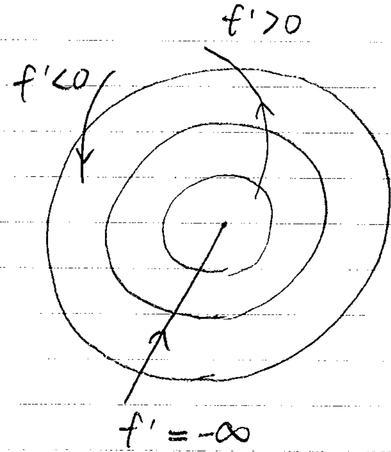
この範囲で $r = a\sqrt{2\cos 2\theta}$

$-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ で考えると

$$\frac{dr}{d\theta} = a \cdot \frac{2(-2\sin 2\theta)}{2\sqrt{2\cos 2\theta}}$$

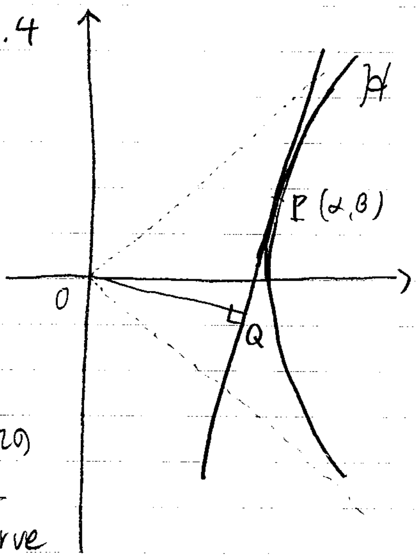
$$= -2a\frac{\sin 2\theta}{\sqrt{2\cos 2\theta}}$$





$\underline{S = 2a^2}$

Task 2.4



Hの原点についての垂足曲線 pedal curve

P(α,β), Q(X,Y) とする

$\ell_P : \alpha x - \beta y = 1$

OQは $\begin{pmatrix}\beta\\ \alpha\end{pmatrix}$ を法線 vector にもつから

$\beta x + \alpha y = 0$

この交点がQ(X,Y)だから $\begin{cases}\alpha X - \beta Y = 1\\ \beta X + \alpha Y = 0\end{cases}$

$$\alpha = \frac{\begin{vmatrix}1 & -Y\\ 0 & X\end{vmatrix}}{\begin{vmatrix}X & -Y\\ Y & X\end{vmatrix}} = \frac{X}{X^2+Y^2}$$

$$\beta = \frac{\begin{vmatrix}X & 1\\ Y & 0\end{vmatrix}}{X^2+Y^2} = \frac{-Y}{X^2+Y^2}$$

P(α,β) ∈ H上の点だから

$$\left(\frac{X}{X^2+Y^2}\right)^2 - \left(\frac{-Y}{X^2+Y^2}\right)^2 = 1$$

$$X^2 - Y^2 = (X^2+Y^2)^2 \quad \text{ただし } (X,Y) \neq (0,0)$$

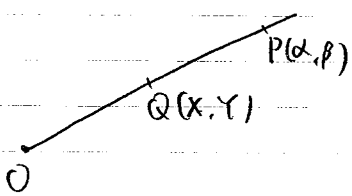
$x = r\cos\theta$, $y = r\sin\theta$ を代入

$r^4 = r^2\cos 2\theta$

$0 \le \theta \le \frac{\pi}{4}$, $\frac{3\pi}{4} \le \theta \le \frac{5\pi}{4}$, $\frac{7\pi}{4} \le \theta \le 2\pi$

$r \neq 0$ より $r^2 = \cos 2\theta$ Lemniscate

cf. Qのx軸鏡映をQ_1とすると、Q_1とPは互いに他の反転である。



Pの $\mathcal{U}$ に関する反転をQ

$OP^2 \cdot OQ^2 = 1$ より $(X^2+Y^2)(\alpha^2+\beta^2) = 1$

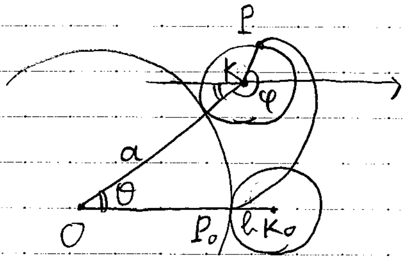
$\vec{OP}$ をX,Yで表す

$$\vec{OP} = |\vec{OP}|\frac{\vec{OQ}}{|\vec{OQ}|} = \frac{\vec{OQ}}{|\vec{OQ}|^2} \text{ より}$$

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{1}{X^2+Y^2}\begin{pmatrix}X\\ Y\end{pmatrix} \qquad \begin{cases}\alpha = \dfrac{X}{X^2+Y^2}\\ \beta = \dfrac{Y}{X^2+Y^2}\end{cases}$$

このx軸鏡映は垂足に一致する

Task 2.5



$a\theta = l\varphi$

$\overrightarrow{OP} = \overrightarrow{OK} + \overrightarrow{KP}$

$\overrightarrow{OK} = (a+l)u_\theta = (a+l)\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$

$\arg \overrightarrow{KP} = -\pi + \theta + \varphi$ より

$$\overrightarrow{KP} = l u_{-\pi+\theta+\varphi} = l\begin{pmatrix}\cos(\theta+\varphi-\pi) \\ \sin(\theta+\varphi-\pi)\end{pmatrix}$$

$$= -l\begin{pmatrix}\cos(\theta+\varphi) \\ \sin(\theta+\varphi)\end{pmatrix}$$

$\therefore P(x, y)$ として

$$\begin{pmatrix}x \\ y\end{pmatrix} = (a+l)\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} - l\begin{pmatrix}\cos(\theta+\varphi) \\ \sin(\theta+\varphi)\end{pmatrix}$$

$\varphi = \dfrac{a}{l}\theta$ より $\theta+\varphi = \dfrac{a+l}{l}\theta$

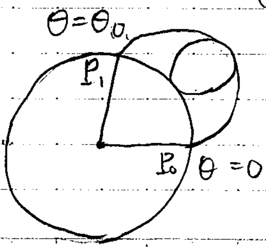
$$\therefore \begin{cases} x = (a+l)\cos\theta - l\cos\dfrac{a+l}{l}\theta \\ y = (a+l)\sin\theta - l\sin\dfrac{a+l}{l}\theta \end{cases}$$

$a+l = A$, $\dfrac{a+l}{l} = B$ とすれば $lB = A$

$$\begin{cases} x = A\cos\theta - l\cos B\theta \\ y = A\sin\theta - l\sin B\theta \end{cases}$$

$$\dot{x} = -A\sin\theta + lB\sin B\theta = -A(\sin\theta - \sin B\theta)$$
$$\dot{y} = A\cos\theta - lB\cos B\theta = A(\cos\theta - \cos B\theta)$$

$$x\,dy = (A\cos\theta - l\cos B\theta)A(\cos\theta - \cos B\theta)\,d\theta$$
$$y\,dx = (A\sin\theta - l\sin B\theta)A(-\sin\theta + \sin B\theta)\,d\theta$$
$$x\,dy - y\,dx = A(A+l)\{1-\cos(B-1)\theta\}d\theta$$
$$= (a+l)(a+2l)\left(1-\cos\frac{a}{l}\theta\right)d\theta$$



$\overset{\frown}{P_0P_1} = a\theta_0 = 2\pi l$ より

$$\theta_0 = \frac{2\pi l}{a}$$

$$S = \frac{1}{2}(a+l)(a+2l)\int_0^{\theta_0}\left(1-\cos\frac{a}{l}\theta\right)d\theta$$

$\cos\frac{a}{l}\theta$の周期は

$$\frac{2\pi}{\frac{a}{l}} = \frac{2\pi l}{a}$$

$$= \frac{1}{2}(a+l)(a+2l)\cdot\frac{2\pi l}{a}$$

おうぎ形は

$$\frac{1}{2}a^2\frac{2\pi l}{a} = \pi a l$$

$$S = \frac{\pi l}{a}(a^2+3al+2l^2) - \pi a l$$

$$= \frac{\pi l^2(3a+2l)}{a}$$