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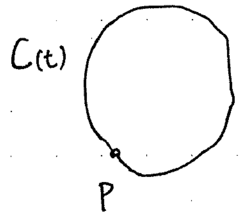
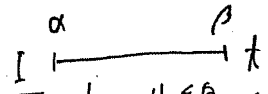
Lecture 1. 曲率

◎ 曲線の分類

∘ 単純閉曲線　simple closed curve



始点と終点が一致し、自己交叉点をもたない

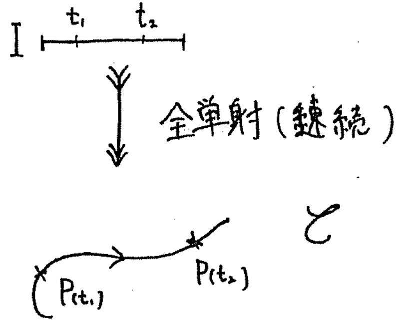


$\mathscr{C}(\alpha) = \mathscr{C}(\beta)$

Jordan 曲線（平面を2つの領域に分ける）

Jordan の閉曲線定理

∘ 単純曲線　有向弧　oriented

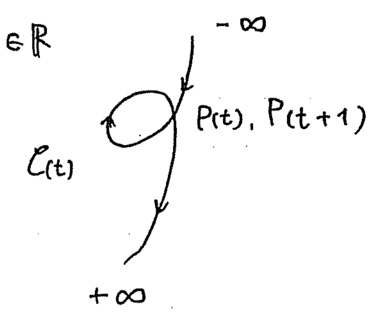


$\mathscr{C}$

$t_1 < t_2 \iff P(t_1) < P(t_2)$ ←-- 数曲線

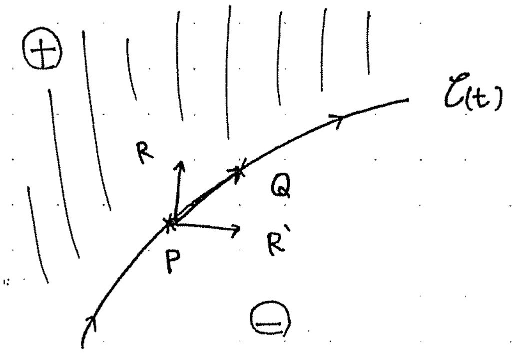
∘ 自己交叉があるとき

$t \in \mathbb{R}$



$\mathscr{C}_1\ (-\infty, t]$
$\mathscr{C}_2\ (t, t+1]$
$\mathscr{C}_3\ (t+1, +\infty)$
ここで
$\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3$: 有向単純弧

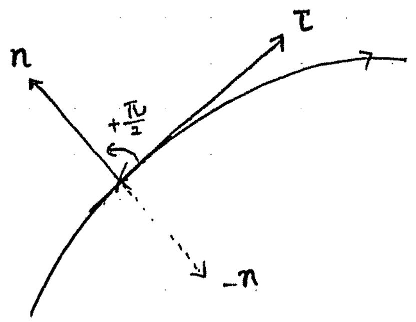
◎ 曲線の正側・負側



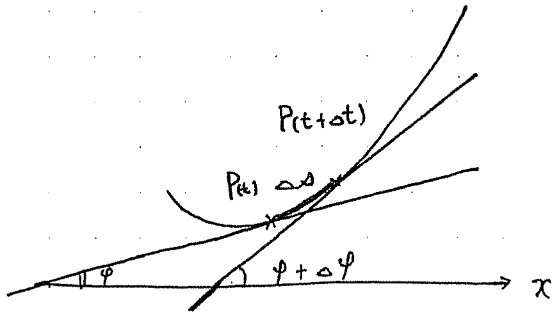
$\overrightarrow{PQ}$ と $\overrightarrow{PR}$ の外積（$\overrightarrow{PQ} \times \overrightarrow{PR}$）が正になる側を正の側といい，（進行方向左）
逆を負の側という．（進行方向右）

正側　left side
負側　right side

∘ 法線・接線の向き



◎ 曲率



弧長の変化を Δs
偏角の 〃 Δφ としたとき

$$\frac{\Delta\varphi}{\Delta s} \xrightarrow[\Delta s \to 0]{\Delta t \to 0} \frac{d\varphi}{ds}$$

これを $\mathscr{C}$ の P における
曲率 curvature κ

$\mathscr{C}(t) : \begin{cases} x = x(t) \\ y = y(t) \end{cases}$ について　$\tan\varphi = \frac{\dot{y}}{\dot{x}}$ より．　$\varphi = \arctan\frac{\dot{y}}{\dot{x}}$

$$\frac{d\varphi}{dt} = \frac{1}{1+\left(\frac{\dot{y}}{\dot{x}}\right)^2} \cdot \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2} = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}$$

$$= \frac{\begin{vmatrix} \dot{x} & \ddot{x} \\ \dot{y} & \ddot{y} \end{vmatrix}}{|\tau|^2}$$

$\frac{ds}{dt} = \sqrt{\dot{x}^2 + \dot{y}^2}$ より．　$\frac{dt}{ds} = \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}}$

よって．　$\kappa = \frac{d\varphi}{ds} = \frac{d\varphi}{dt} \cdot \frac{dt}{ds}$

$$= \frac{\begin{vmatrix} \dot{x} & \ddot{x} \\ \dot{y} & \ddot{y} \end{vmatrix}}{|\tau|^3} = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}$$

単位円 $\mathcal{U}$ について．$\kappa = 1$
半径 a の円　$\kappa = \frac{1}{a}$

曲率半径

$\mathscr{C}(t)$ のある点における曲率が κ のとき．
その点における曲率半径　$\rho = \frac{1}{|\kappa|}$

Problems 1

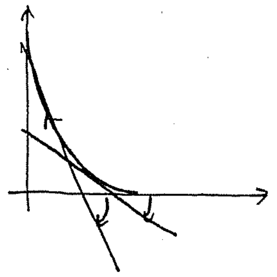
$\boxed{\text{Task 1-1}}$

(2) Astroid　$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$　$\mathscr{C}$.

$x^{\frac{1}{3}} = \cos\theta$．　$y^{\frac{1}{3}} = \sin\theta$ として．
$\mathscr{C}$ を para-表示 $\begin{cases} x = \cos^3\theta \\ y = \sin^3\theta \end{cases}$

$\dot{x} = -3\cos^2\theta \cdot \sin\theta$　　$\ddot{x} = -3(\cos^3\theta - 2\sin^2\theta \cdot \cos\theta) = -3\cos\theta(\cos^2\theta - 2\sin^2\theta)$
$\dot{y} = 3\sin^2\theta \cdot \cos\theta$　　$\ddot{y} = 3\sin\theta(2\cos^2\theta - \sin^2\theta)$

$$\therefore \begin{vmatrix} \dot{x} & \ddot{x} \\ \dot{y} & \ddot{y} \end{vmatrix} = \dot{x}\ddot{y} - \ddot{x}\dot{y}$$
$$= \cdots = -9\sin^2\theta\cos^2\theta \quad \underline{(< 0)}$$

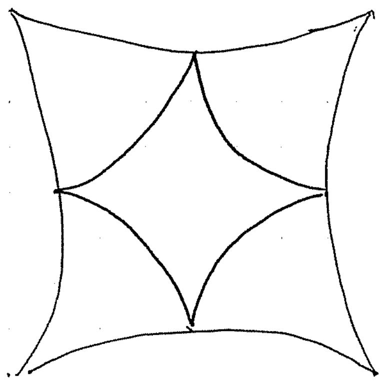


−30° くらいから −45° くらいに！
曲率負の値！

分母　$\dot{x}^2 + \dot{y}^2 = 9\sin^2\theta \cdot \cos^2\theta$ より．
$(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}} = 27\sin^3\theta \cdot \cos^3\theta$

$$\therefore \kappa = \frac{-1}{3\sin\theta \cdot \cos\theta}$$

よって．
$$\rho = 3|\sin\theta \cdot \cos\theta|$$

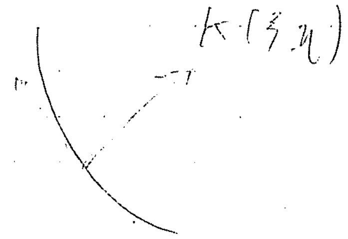


曲率中心の軌跡も
2倍拡大して $\frac{\pi}{4}$ 回転した
Astroid!!

曲率中心の軌跡 $\mathcal{L}(K)$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}-3\sin\theta\cdot\cos^2\theta\\ 3\sin^2\theta\cdot\cos\theta\end{pmatrix} = 3\sin\theta\cdot\cos\theta\begin{pmatrix}-\cos\theta\\ \sin\theta\end{pmatrix}$$

$\theta \neq 0, \frac{\pi}{2}$ ならば 接vector $\tau = \begin{pmatrix}-\cos\theta\\ \sin\theta\end{pmatrix}$



$K(\xi,\eta)$ として.

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\cos^3\theta\\ \sin^3\theta\end{pmatrix} + 3\sin\theta\cdot\cos\theta\begin{pmatrix}\sin\theta\\ \cos\theta\end{pmatrix}$$

$$\xi = \cos^3\theta + 3\cos\theta\cdot\sin^2\theta$$
$$\eta = 3\cos^2\theta\cdot\sin\theta + \sin^3\theta$$

$$\underbrace{(\xi+\eta)}_{X}{}^{\frac{2}{3}} + \underbrace{(\xi-\eta)}_{Y}{}^{\frac{2}{3}} =$$

曲率円は.

$$(x-\xi)^2 + (y-\eta)^2 = 9\sin^2\theta\cdot\cos^2\theta.$$

$\mathcal{E}$を追跡する.

$\xi+\eta = (\cos\theta+\sin\theta)^3$, $\xi-\eta = (\cos\theta-\sin\theta)^3$

$\mathcal{E}$を$\frac{\pi}{4}$回転.

$$R_{\frac{\pi}{4}}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}\xi-\eta\\ \xi+\eta\end{pmatrix}$$

$\frac{1}{\sqrt{2}}(\xi-\eta) = \overline{x}$, $\frac{1}{\sqrt{2}}(\xi-\eta) = \overline{y}$ とすれば.

$\xi-\eta = \sqrt{2}\,\overline{x}$, $\xi+\eta = \sqrt{2}\,\overline{y}$

$$\therefore (\xi+\eta)^{\frac{2}{3}} + (\xi-\eta)^{\frac{2}{3}} = (\cos\theta+\sin\theta)^2 + (\cos\theta-\sin\theta)^2 = 2$$

$$\therefore (2^{\frac{1}{2}}\overline{x})^{\frac{2}{3}} + (2^{\frac{1}{2}}\overline{y})^{\frac{2}{3}} = 2$$

$$\therefore \overline{x}^{\frac{2}{3}} + \overline{y}^{\frac{2}{3}} = 2^{\frac{2}{3}}$$

車輪 トロコス → ころがるトロッコ.
円を転がした時の端点等の軌跡 → トロコイド

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(3) $\mathcal{C}_3 : \begin{cases} x = a(t-\sin t)\\ y = a(1-\cos t)\end{cases}$ Cycloid

簡単のためa=1とする.

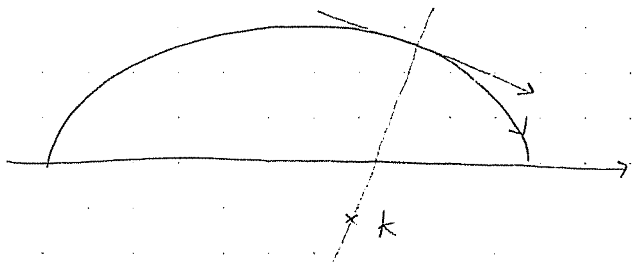
$\dot{x} = 1-\cos t$ $\dot{y} = \sin t$
$\ddot{x} = \sin t$ $\ddot{y} = \cos t$

$$\dot{x}^2+\dot{y}^2 = 1-2\cos t+1 = 2(1-\cos t)$$
$$\dot{x}\ddot{y}-\ddot{x}\dot{y} = \cos t(1-\cos t)-\sin^2 t$$
$$= \cos t - 1$$

$$\therefore \kappa = \frac{-(1-\cos t)}{\{2(1-\cos t)\}^{\frac{3}{2}}} = \frac{-1}{2^{\frac{3}{2}}\cdot(1-\cos t)^{\frac{1}{2}}}$$
$$= \frac{-1}{2^{\frac{3}{2}}\cdot|2^{\frac{1}{2}}\sin\frac{t}{2}|} = \frac{-1}{4|\sin\frac{t}{2}|}$$

よって.

$$\rho = 4|\sin\tfrac{t}{2}|$$



接vector

$$\tau = \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}1-\cos t\\ \sin t\end{pmatrix}$$
$$\therefore n = \begin{pmatrix}\sin t\\ \cos t - 1\end{pmatrix}$$

$$|n|^2 = 2-2\cos t = 2(1-\cos t)$$
$$= 4\sin^2\tfrac{t}{2}$$
$$\therefore |n| = 2|\sin\tfrac{t}{2}|$$

$$\therefore \overrightarrow{OK} = \overrightarrow{OT} + \overrightarrow{TK}$$
$$= \begin{pmatrix}t-\sin t\\ 1-\cos t\end{pmatrix} + \rho\cdot\frac{n}{|n|}$$
$$= \begin{pmatrix}t-\sin t\\ 1-\cos t\end{pmatrix} + 2\begin{pmatrix}\sin t\\ \cos t - 1\end{pmatrix}$$
$$= \begin{pmatrix}t+\sin t\\ \cos t - 1\end{pmatrix}$$

よって. $\mathcal{E} : \begin{cases} \xi = t+\sin t\\ \eta = \cos t - 1\end{cases}$

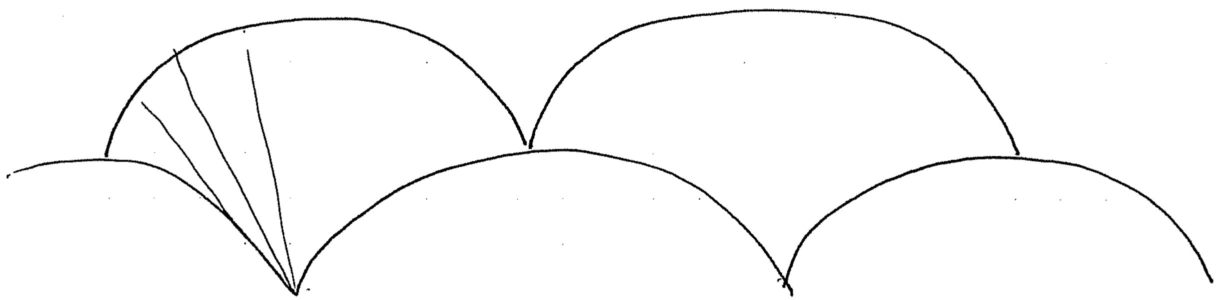
変換してみる.

$$\begin{cases} \xi = \{(t+\pi) - \sin(t+\pi)\} - \pi \\ \eta = 1 - \cos(t+\pi) - 2 \end{cases}$$

$$\Leftrightarrow \quad \begin{aligned} \xi + \pi &= (t+\pi) - \sin(t+\pi) \\ \eta + 2 &= 1 - \cos(t+\pi) \end{aligned}$$

つまり.

$\mathcal{E}$ は $\mathcal{C}_3$ を $\begin{cases} x\text{方向} & -\pi \\ y\text{方向} & -2 \end{cases}$ 平行移動したもの.



等時ふり子

1/4 終了…

$\boxed{\text{Task 1-2}}$ $r = f(\theta)$ (極座標表示)

(1) $\rho = \dfrac{(r^2 + \dot r^2)^{\frac{3}{2}}}{|r^2 + 2\dot r^2 - r\ddot r|}$ ⌋ SHOW!

$\mathcal{C}$ を para-表示して

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = R(\theta)\begin{pmatrix} r \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \dot x \\ \dot y \end{pmatrix} = R(\theta + \tfrac{\pi}{2})\begin{pmatrix} r \\ 0 \end{pmatrix} + R(\theta)\begin{pmatrix} \dot r \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \ddot x \\ \ddot y \end{pmatrix} = R(\theta+\pi)\begin{pmatrix} r \\ 0 \end{pmatrix} + 2R(\theta + \tfrac{\pi}{2})\begin{pmatrix} \dot r \\ 0 \end{pmatrix} + R(\theta$$

$$\dot{\mathbf{x}} \times \ddot{\mathbf{x}} = \left[\begin{pmatrix} 0 \\ r \end{pmatrix} + \begin{pmatrix} \dot r \\ 0 \end{pmatrix}\right] \times \left[\begin{pmatrix} -r \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2\dot r \end{pmatrix} + \begin{pmatrix} \ddot r \\ 0 \end{pmatrix}\right]$$

$$\begin{cases} \dot x = \dot r\cos\theta + r(-\sin\theta) \\ \dot y = \dot r\sin\theta + r\cos\theta \end{cases}$$

$$\ddot x = \ddot r\cos\theta + 2\dot r(-\sin\theta) + r(-\cos\theta)$$
$$\ddot y = \ddot r\sin\theta + 2\dot r\cos\theta + r(-\sin\theta)$$

曲率 κ について分子

$$\begin{vmatrix} \dot x & \ddot x \\ \dot y & \ddot y \end{vmatrix} = r^2 - r\ddot r + 2\dot r^2$$

$$\dot x^2 + \dot y^2 = \dot r^2 + r^2$$

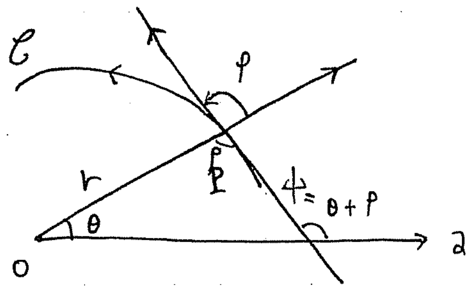
よって,

$$\kappa = \frac{r^2 + 2\dot r^2 - r\ddot r}{\{r^2 + \dot r^2\}^{\frac{3}{2}}}$$

だから,

$$\rho = \frac{\{r^2 + \dot r^2\}^{\frac{3}{2}}}{|r^2 + 2\dot r^2 - r\ddot r|}$$

(2)



$\tan\rho = \dfrac{r}{\dot r}$ ⌋ SHOW!

$$\frac{x\dot y - \dot x y}{x\dot x + y\dot y} = \frac{r \times \dot r}{r \cdot \dot r}$$

$\theta + \rho = \psi$ より $\varphi = \psi - \theta$

$$\tan\varphi = \tan(\psi - \theta)$$
$$= \frac{\tan\psi - \tan\theta}{1 + \tan\psi \cdot \tan\theta} = \frac{\dot y/\dot x - \tan\theta}{1 + (\dot y/\dot x)\cdot\tan\theta} \qquad \left(\because \tan\psi = \frac{\dot y}{\dot x}\right)$$

ここで $\dfrac{\dot y}{\dot x} = \dfrac{\dot r\sin\theta + r\cos\theta}{\dot r\cos\theta - r\sin\theta}$

$\cos\theta \neq 0$ の下で.

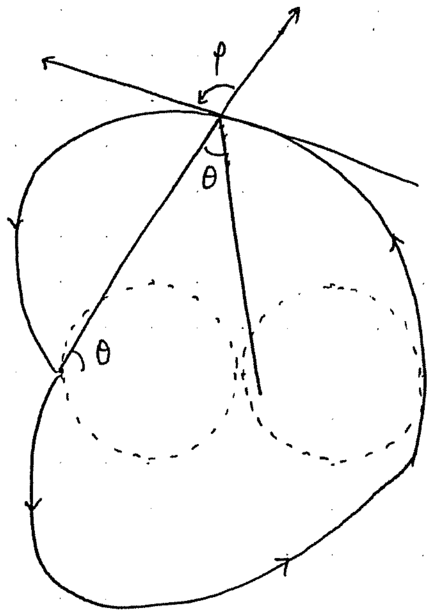
$$\frac{\dot{y}}{\dot{x}} = \frac{\dot{r}\tan\theta + r}{\dot{r} - r\tan\theta}$$

$$\therefore \tan\varphi = \frac{\frac{\dot{r}\tan\theta + r}{\dot{r} - r\tan\theta} - \tan\theta}{1 + \frac{\dot{r}\tan\theta + r}{\dot{r} - r\tan\theta}\cdot\tan\theta}$$

$$= \frac{r + r\tan^2\theta}{\dot{r} + \dot{r}\tan^2\theta} = \frac{r(1+\tan^2\theta)}{\dot{r}(1+\tan^2\theta)}$$

$$= \frac{r}{\dot{r}}$$

(3)



$\mathcal{C}$: Cardioid

$r = 1 + \cos\theta$

動径と接線のなす角を φ として.

$$\tan\varphi = \frac{r}{\dot{r}} = \frac{1+\cos\theta}{-\sin\theta}$$

$$= -\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}} = -\frac{1}{\tan\frac{\theta}{2}}$$

$$= \tan\left(\frac{\theta}{2} + \frac{\pi}{2}\right)$$

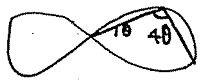
入・反射角はいずれも $\pi - \varphi$
よって光線のなす角は

$$\pi - 2(\pi - \varphi) = 2\varphi - \pi$$

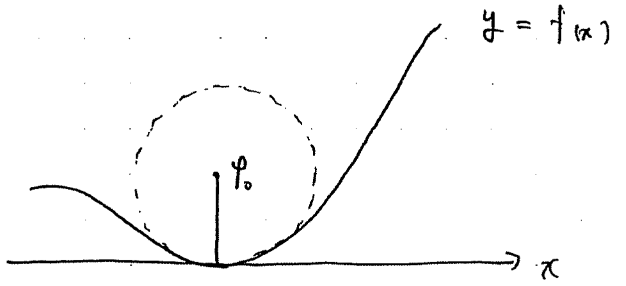
$$= 2\left(\frac{\pi}{2} + \frac{\theta}{2}\right) - \pi$$

$$= \theta$$

cf. レムニスケート



$$\boxed{\text{Task 1 - 3}}$$



$$\rho_0 = \lim_{x\to 0}\left|\frac{x^2}{2y}\right|$$ SHOW!

f が2階 contin. dif'ble だから.
f, f', f'' は contin.
$y = f(x)$ の曲率 K は.

$$K = \frac{\ddot{y}}{(1+\dot{y}^2)^{\frac{3}{2}}} = \frac{f''(x)}{\{1+f'^2(x)\}^{\frac{3}{2}}}$$

よって.

$$\rho = \frac{\{1 + f'(x)^2\}^{\frac{3}{2}}}{|f''(x)|}$$

$\mathcal{C}$ は原点 O で x 軸に接するから. $f'(0) = 0$

$$\therefore \rho_{x=0} = \frac{1}{|f''(0)|}$$

$f(x)$ についての仮定から.

$$y = f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(\theta x)}{2!}x^2 \quad \theta \in (0.1)$$

$$= \frac{f''(\theta x)}{2!}x^2$$

$$\therefore f''(\theta x) = \frac{2y}{x^2}$$

f'' は contin だから.

$$f''(0) = \lim_{x\to 0} f''(\theta x) = \lim_{x\to 0}\frac{2y}{x^2}$$

$$\therefore \rho_0 = \frac{1}{|f''(0)|} = \lim_{x\to 0}\left|\frac{x^2}{2y}\right|$$

cf. 上から目線1

$1+kx^2 \leq \cos x \iff 1-\cos x \leq -kx^2$

$-k=l$ と書けば l の範囲を求めれば十分.

$l>0$ が必要.

$y=1-\cos x=f(x)$, $y=lx^2=g(x)$ とすると.

f, g は仮定をみたすから.

曲率半径 ρ^f, ρ^g として.

$$\rho^f_{x=0} = \lim_{x\to 0}\left|\frac{x^2}{2y}\right| = \lim_{x\to 0}\frac{x^2}{2(1-\cos x)} = 1$$

また,

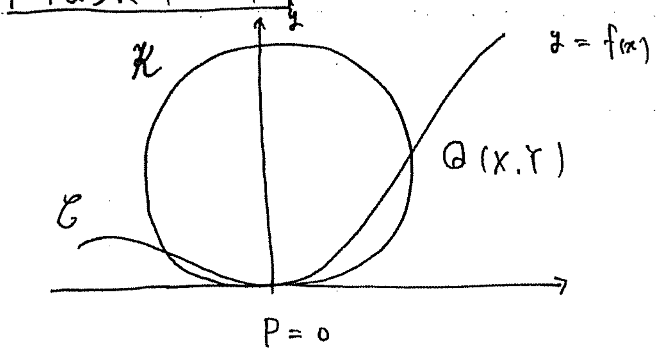
$$\rho^g_{x=0} = \lim_{x\to 0}\left|\frac{x^2}{2y}\right| = \frac{1}{2l}$$

$\rho^f_{x=0} \geq \rho^g_{x=0}$ より.

$$1 \geq \frac{1}{2l} \qquad \therefore\ l \geq \frac{1}{2}$$

$$\therefore\ \underline{k \leq -\frac{1}{2}}_{\,/\!/}$$

Task 1-4



$\mathcal{C}$, $\mathcal{K}$ の接点Pを原点 $\overline{O}$ に.
接線を x 軸に重ねる.

$\mathcal{C}$ が原点の近傍で $y=f(x)$ と表されるとする.

$$\kappa = \frac{\ddot{y}}{(1+\dot{y}^2)^{\frac{3}{2}}} \text{ より.}$$

$$\rho = \frac{(1+\dot{y}^2)^{\frac{3}{2}}}{|\ddot{y}|}$$

$x=0$ とすれば.

$$\rho_{x=0} = \frac{1}{|\ddot{y}|}$$

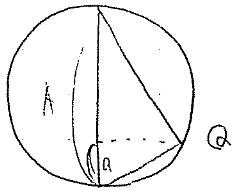
円 K の中心は y 軸上にあるから.
半径 r として 中心 $(0, r)$

$$\therefore\ K: x^2+(y-r)^2=r^2$$

$\mathcal{C}$, $\mathcal{K}$ の交点を $Q(X, Y)$ とすれば

$$Y=f(X),\quad X^2+(Y-r)^2=r^2$$

$$r=\frac{X^2+Y^2}{2Y}$$



$\underset{OQ^2}{\underline{X^2+Y^2}} = aA$

$Q \to P$ のとき $X \to 0$ だから.

$r=\dfrac{X^2+Y^2}{2Y}$ の $X\to 0$ のときの極限を考える.

$X\to 0$ のとき. $Y\to 0$, $\dot{Y}\to 0$.

$$\lim_{X\to 0}\frac{X^2+Y^2}{2Y} = \lim_{X\to 0}\frac{2X+2Y\cdot\dot{Y}}{2\dot{Y}}$$

$$= \lim_{X\to 0}\frac{1+\dot{Y}^2+Y\ddot{Y}}{\ddot{Y}} = \frac{1}{\ddot{Y}_{X=0}} \qquad (\because \text{ロピタル})$$

よって.

$$r \xrightarrow[X\to 0]{} \frac{1}{\ddot{Y}_{X=0}} = \frac{1}{f''(0)} = \rho_{x=0}$$

上サシ目線2

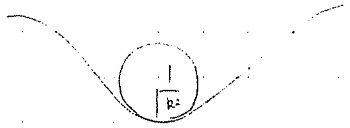
$\mathcal{C} : y = \cos kx$

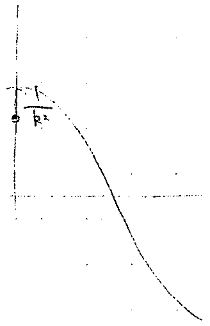
$\mathcal{C}_1$ として $y = 1 - \cos kx$ を考えると，
仮定をみたす.

$\dot{y} = k \sin kx \quad , \ \ddot{y} = k^2 \cos kx$

$\therefore \ \ddot{y}_{x=0} = k^2$

$\therefore \ \rho_{x=0} = \dfrac{1}{k^2}$





3点通る円 ⟺ 接線共有，2点通る.

$\boxed{\text{Task 1-5}}$ $\ \mathcal{E} : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases} \qquad \begin{cases} \dot{x} = -a\sin t \\ \dot{y} = b\cos t \end{cases} \qquad \begin{cases} \ddot{x} = -a\cos t \\ \ddot{y} = -b\sin t \end{cases}$$

$$\dot{x}^2 + \dot{y}^2 = a^2\sin^2 t + b^2\cos^2 t$$
$$\dot{x}\ddot{y} - \ddot{x}\dot{y} = ab$$

よって，

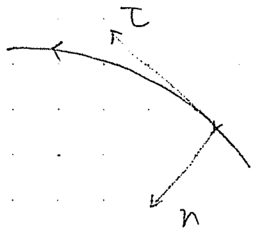
$$\kappa = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}}$$

$$= \frac{a^4 b^4}{(a^4 b^2 \sin^2 t + a^2 b^4 \cos^2 t)^{\frac{3}{2}}}$$

$$= \frac{a^4 b^4}{(a^4 y^2 + b^4 x^2)^{\frac{3}{2}}} \qquad (>0)$$

$$\therefore \rho = \frac{(b^4 x^2 + a^4 y^2)^{\frac{3}{2}}}{a^4 b^4} \quad \left(= \frac{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}}{ab} \right)$$

曲率中心



接 vector $\tau = \begin{pmatrix} -a\sin t \\ b\cos t \end{pmatrix}$

法線 vector $n = \begin{pmatrix} -b\cos t \\ -a\sin t \end{pmatrix}$

$|n|^2 = a^2\sin^2 t + b^2\cos^2 t$

$\therefore \overrightarrow{TK} = \rho \dfrac{n}{|n|} = \dfrac{a^2\sin^2 t + b^2\cos^2 t}{ab} \begin{pmatrix} -b\cos t \\ -a\sin t \end{pmatrix}$

$$\overrightarrow{OK} = \overrightarrow{OT} + \overrightarrow{TK} = \begin{pmatrix} a\cos t \\ b\sin t \end{pmatrix} + \frac{a^2\sin^2 t + b^2\cos^2 t}{ab} \begin{pmatrix} -b\cos t \\ -a\sin t \end{pmatrix}$$

$$\xi = a\cos t - \frac{a^2\sin^2 t + b^2\cos^2 t}{ab} b\cos t$$
$$= \frac{1}{ab}(a^2 b\cos t - a^2 b\sin^2 t\cos t - b^3\cos^3 t)$$
$$= \frac{1}{a}(a^2\cos^3 t - b^2\cos^3 t) = \frac{a^2 - b^2}{a}\cos^3 t$$

$$\eta = -\frac{a^2 - b^2}{b}\sin^3 t$$

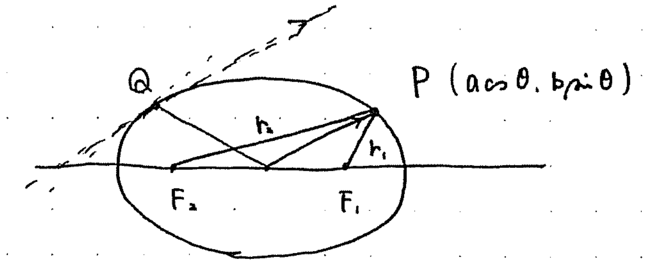
$\cos^3 t = \frac{a}{a^2-b^2}\xi$ より.

$\cos^2 t = \left(\frac{a}{a^2-b^2}\xi\right)^{\frac{2}{3}}$

$\sin^2 t = \left(-\frac{b}{a^2-b^2}\eta\right)^{\frac{2}{3}}$

$\therefore \left(\frac{a}{a^2-b^2}\xi\right)^{\frac{2}{3}} + \left(-\frac{b}{a^2-b^2}\eta\right)^{\frac{2}{3}} = 1$

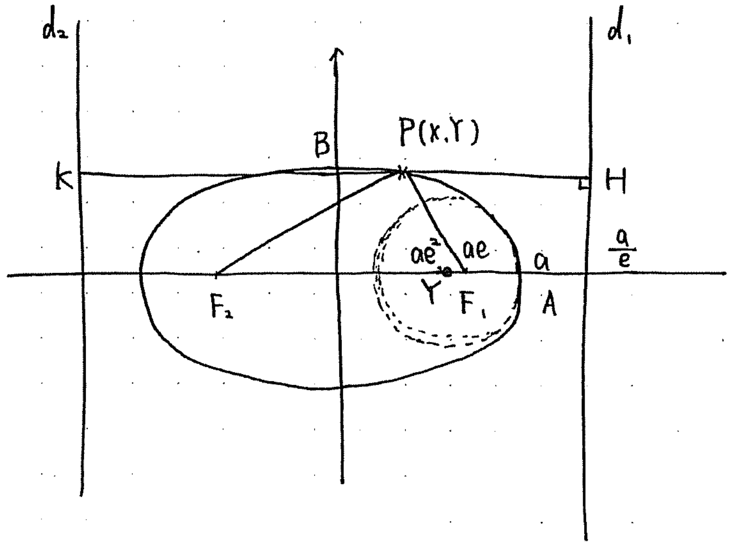
共役半径定理



OQ を OP の共役半径.

$\overrightarrow{OP} = \begin{pmatrix} a\cos\theta \\ b\sin\theta \end{pmatrix}$ ならば $\overrightarrow{OQ} = \begin{pmatrix} a\cos(\theta+\frac{\pi}{2}) \\ b\sin(\theta+\frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} -a\sin\theta \\ b\cos\theta \end{pmatrix}$

$\therefore OQ^2 = a^2\sin^2\theta + b^2\cos^2\theta = \frac{1}{a^2b^2}(a^4y^2 + b^4x^2)$



$r_1 = PF_1 = e\,PH = e\left(\frac{a}{e} - x\right) = a - eX$

$r_2 = PF_2 = e \cdot PK = a + eX$

$r_1 + r_2 = 2a$

$r_1 r_2 = a^2 - e^2X^2$

$= a^2(1 - e^2\cos^2\theta)$

また.

$OQ^2 = \quad (\sin^2\theta + \frac{b^2}{a^2}\cos^2\theta)$

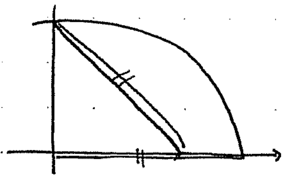
$= a^2\left(1 - \frac{a^2-b^2}{a^2}\cos^2\theta\right)$

$= a^2(1 - e^2\cos^2\theta)$

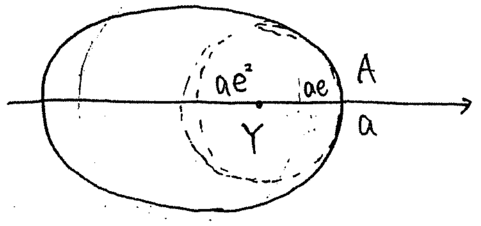
$\therefore r_1 r_2 = OQ^2$

$a^2e^2 + b^2 = a^2$

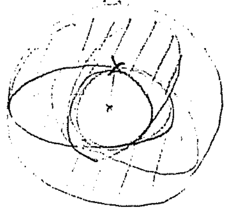
$\therefore e^2 = \frac{a^2-b^2}{a^2}$

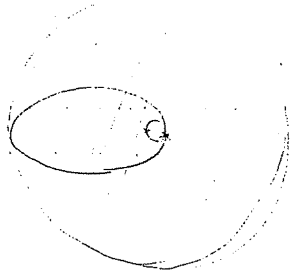


$\rho = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}}{ab} = \frac{OQ^3}{ab} = \underline{\frac{(r_1 r_2)^{\frac{3}{2}}}{ab}}\ /\!/$



cf.





$t = 0$ のとき $\xi = \frac{a^2-b^2}{a} = \frac{a^2e^2}{a} = ae^2$

Y は補助円に関する A の反転

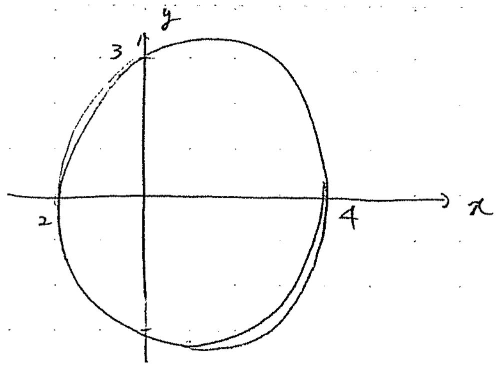
Task 2-1

(1) $r = 3 + \cos\theta$

まず概形. $r(0) = 4$

$r(\pm\frac{\pi}{2}) = 3$

$r(\pi) = 2$



面積要素 $dS = \frac{1}{2}r^2 \cdot d\theta$

$$S = 2\int_{\theta=0}^{\theta=\pi} dS$$
$$= \int_0^{\pi} r^2 d\theta$$
$$= \int_0^{\pi} (9 + 6\cos\theta + \cos^2\theta)\, d\theta$$
$$= 9\pi + 0 + \frac{\pi}{2} = \underline{\frac{19}{2}\pi} /\!/$$

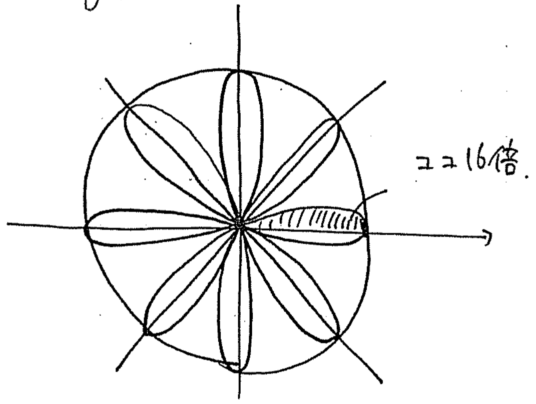
(2) $r = \cos 4\theta = g(\theta)$

$$g(\frac{\pi}{2} - \theta) = \cos 4(\frac{\pi}{2} - \theta)$$
$$= \cos(2\pi - 4\theta)$$
$$= \cos 4\theta = g(\theta)$$

$$g(-\theta) = \cos(-4\theta) = \cos 4\theta = g(\theta)$$

$\theta = \frac{\pi}{4}$ についても対称.

始線 〃



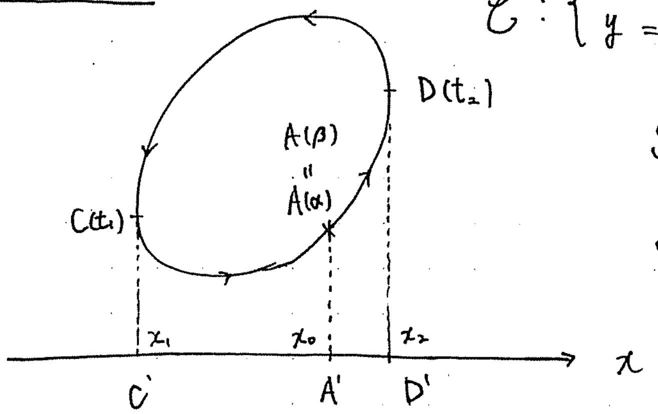
$$dS = \frac{1}{2}r^2 d\theta$$
$$= \frac{1}{2}\cos^2 4\theta \cdot d\theta$$

$$S = 16\int_0^{\frac{\pi}{8}} \frac{1}{2}\cos^2 4\theta\, d\theta$$
$$= 8\int_0^{\frac{\pi}{8}} \cos^2 4\theta \cdot d\theta$$
$$= 8 \cdot \frac{\pi}{16} = \frac{\pi}{2}$$

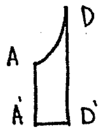
Fundamental Formula of Area

(i) xでint.

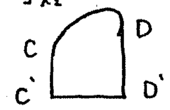
$$\mathscr{C} : \begin{cases} x = x(t) \\ y = y(t) \end{cases}$$



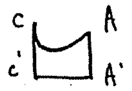
$$S_1 = S_{AA'D'D} = \int_{x_0}^{x_2} y\, dx = \int_{\alpha}^{t_2} y \cdot \frac{dx}{dt} \cdot dt$$



$$S_2 = \qquad = \int_{x_1}^{x_2} y\, dx = -\int_{x_2}^{x_1} y\, dx$$
$$= \int_{t_2}^{t_1} y \frac{dx}{dt} \cdot dt$$

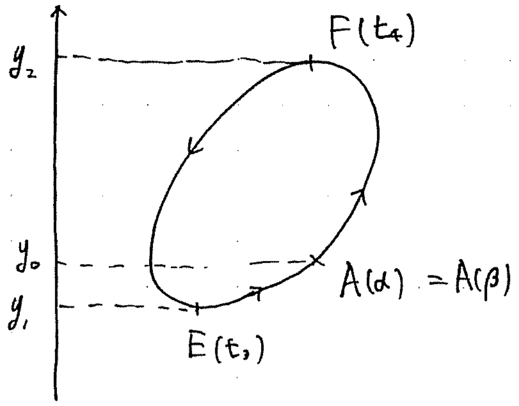


$$S_3 = S_{CC'A'A} = \int_{x_1}^{x_0} y\, dx$$
$$= \int_{t_1}^{\beta} y \cdot \frac{dx}{dt} \cdot dt$$



$$S = S_2 - S_1 - S_3 = -\int_{\alpha}^{t_2} - \int_{t_2}^{t_1} - \int_{t_1}^{\beta}$$
$$= -\int_{\alpha}^{\beta} y \cdot \frac{dx}{dt} \cdot dt \quad \cdots ①$$

(ii) yでint.



$$S = \int_{y_0}^{y_2} + \int_{y_1}^{y_0} - \int_{y_1}^{y_2}$$
$$= \int_{\alpha}^{t_4} + \int_{t_3}^{\beta} - \int_{t_3}^{t_4}$$
$$= \int_{\alpha}^{t_4} + \int_{t_4}^{t_3} + \int_{t_3}^{\beta} = \int_{\alpha}^{\beta}$$
$$= \int_{\alpha}^{\beta} x \cdot \frac{dy}{dt} \cdot dt \quad \cdots ②$$

① と ② を合わせて.

$$S = \frac{1}{2}\int_{\alpha}^{\beta} (x\dot{y} - \dot{x}y) dt$$
$$= \frac{1}{2}\int_{\alpha}^{\beta} \underline{(x dy - y dx)}$$

cf. これは交代式である.

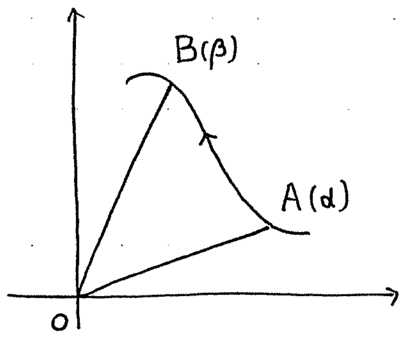
また、$x\dot{y} - \dot{x}y$ について

$$\frac{d}{dt}\left(\frac{y}{x}\right) = \frac{x\dot{y} - \dot{x}y}{x^2}$$ より.

$$x\dot{y} - \dot{x}y = x^2 \cdot \frac{d}{dt}\left(\frac{y}{x}\right)$$

$$\therefore\ S = \frac{1}{2}\int_\alpha^\beta x^2 \left\{\frac{d}{dt}\left(\frac{y}{x}\right)\right\} \cdot dt$$

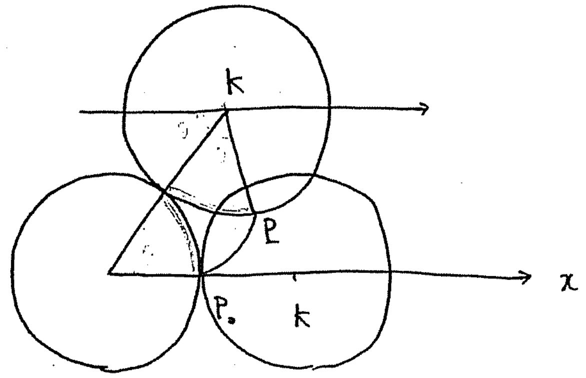
☆ 広義の扇形の面積



$$\mathcal{C}_1 \begin{cases} t : \alpha-1 \to \alpha & OA \\ t : \alpha \to \beta & \overset{\frown}{AB} \\ t : \beta \to \beta+1 & BO \end{cases}$$ とすれば.

$$\begin{aligned} S &= \frac{1}{2}\int_{\alpha-1}^{\beta+1} \\ &= \frac{1}{2}\left[\int_{\alpha-1}^{\alpha} x^2 \cdot \overset{\text{直線}}{\underbrace{\left(\frac{d}{dt}\frac{y}{x}\right)}_{0}} dt + \int_\alpha^\beta + \int_\beta^{\beta+1} x^2 \cdot \overset{\text{傾き一定}}{\underbrace{\left(\frac{d}{dt}\frac{y}{x}\right)}_{0}} dt\right] \\ &= \frac{1}{2}\int_\alpha^\beta \end{aligned}$$

Task 2-2



$\arg \overrightarrow{KP} = \pi + 2\theta$ より.

$$\overrightarrow{KP} = \begin{pmatrix} \cos(\pi+2\theta) \\ \sin(\pi+2\theta) \end{pmatrix} = \begin{pmatrix} -\cos 2\theta \\ -\sin 2\theta \end{pmatrix}$$

また.

$$\overrightarrow{OK} = 2\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$ より.

$$\overrightarrow{OP} = \begin{pmatrix} 2\cos\theta - \cos 2\theta \\ 2\sin\theta - \sin 2\theta \end{pmatrix}$$

$$\therefore \begin{cases} x = 2\cos\theta - \cos 2\theta \\ y = 2\sin\theta - \sin 2\theta \end{cases}$$

(*) 極表示

$$x = 2\cos\theta - (2\cos^2\theta - 1)$$
$$x - 1 = 2\cos\theta(1 - \cos\theta)$$
$$\begin{aligned} y &= 2\sin\theta - 2\sin\theta\cos\theta \\ &= 2\sin\theta(1 - \cos\theta) \end{aligned}$$

$$\therefore \begin{pmatrix} x-1 \\ y \end{pmatrix} = 2(1-\cos\theta)\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\therefore\ r = 2(1 - \cos\theta)$$ （← (1.0) を始点とした極表示）

$dS = \frac{1}{2}r^2 d\theta$ でも求められる.

$$\begin{aligned} x\,dy &= (2\cos\theta - \cos 2\theta)(2\cos\theta - 2\cos 2\theta)\,d\theta \\ &= (4\cos^2\theta - 6\cos\theta\cdot\cos 2\theta + 2\cos^2 2\theta)\,d\theta \end{aligned}$$

$$\begin{aligned} y\,dx &= (2\sin\theta - \sin 2\theta)(-2\sin\theta + 2\sin 2\theta)\,d\theta \\ &= -(4\sin^2\theta - 6\sin\theta\sin 2\theta + 2\sin^2 2\theta)\,d\theta \end{aligned}$$

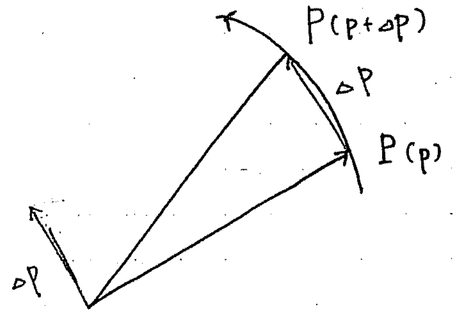
$$\begin{aligned} \therefore\ x\,dy - y\,dx &= [6 - 6(\cos\theta\cos 2\theta + \sin\theta\sin 2\theta)]\,d\theta \\ &= 6(1 - \cos\theta)\,d\theta \end{aligned}$$

$$\begin{aligned} S &= \frac{1}{2}\int_{\theta=0}^{\theta=2\pi} 6(1 - \cos\theta)\,d\theta \\ &= \underline{6\pi} \,/\!/ \end{aligned}$$

M3Ω - ㉑

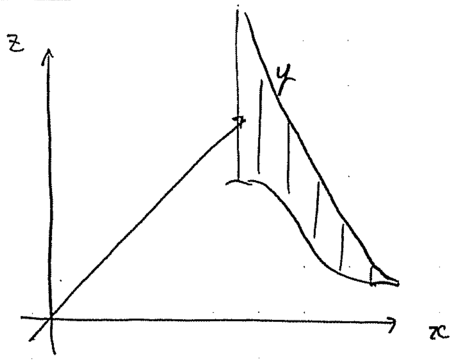
2011. 10 2

cf. Fundamental Formula of Area の意味



$$2\triangle S = \left[P \times \triangle P \right]_z$$
$$= \left[\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \times \begin{pmatrix} \triangle x \\ \triangle y \\ 0 \end{pmatrix} \right]_z$$
$$= \left[\begin{pmatrix} 0 \\ 0 \\ x\triangle y - y\triangle x \end{pmatrix} \right]_z = x\triangle y - y\triangle x \quad \left(= \left(x\frac{\triangle y}{\triangle t} - y\frac{\triangle x}{\triangle t} \right)\triangle t \right)$$

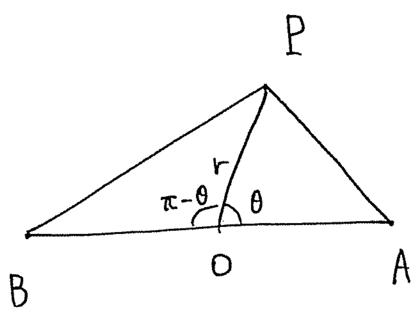
★イメージ



壁を立てていく.イメージ
(高さ = 面積)

Task 2-3 極座標と余弦定理は相性が良い!!

M3Ω - ㉒



$$\begin{cases} PA \cdot PB = a^2 \\ AB = 2a \end{cases}$$

P の軌跡 $\mathcal{L}(\underline{P})$
とその囲む面積 S」FIND!

$$PA^2 = r^2 + a^2 - 2ra\cos\theta$$
$$PB^2 = r^2 + a^2 - 2ra\cos(\pi - \theta)$$
$$= r^2 + a^2 + 2ra\cos\theta$$

$$PA^2 \cdot PB^2 = (r^2 + a^2)^2 - 4r^2a^2\cos^2\theta = a^4$$

$$\therefore r^4 + 2a^2r^2 - 4r^2a^2\cos^2\theta = 0.$$

$r \neq 0$ の下.

$$r^2 = 2a^2(2\cos^2\theta - 1)$$
$$= 2a^2\cos 2\theta$$

$r = 0$ のとき. $\underline{P} = \overline{0}$

$$r = a\sqrt{2\cos 2\theta}$$

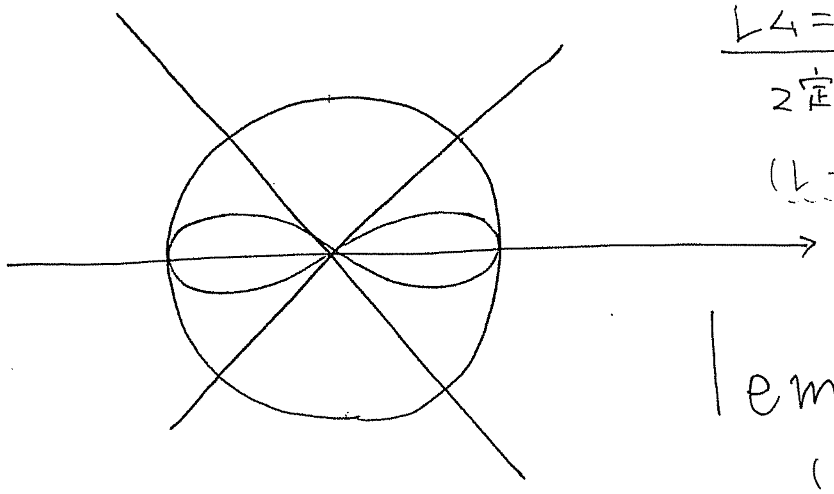
$-\pi < \theta \leq \pi \longrightarrow -2\pi < \theta \leq 2\pi$

$\cos 2\theta \geq 0$ だから.

$$-2\pi < 2\theta \leq -\frac{3\pi}{2}$$
$$-\frac{\pi}{2} \leq 2\theta \leq \frac{\pi}{2}$$
$$\frac{3\pi}{2} \leq 2\theta \leq 2\pi$$

$\Longleftrightarrow$

$$-\pi < \theta \leq -\frac{3\pi}{4}$$
$$-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$$
$$\frac{3\pi}{4} \leq \theta \leq \pi$$



レムニスケート!

2定点からのきょりの積が一定

(レームニスコス:リボン形)
ギリシャ語

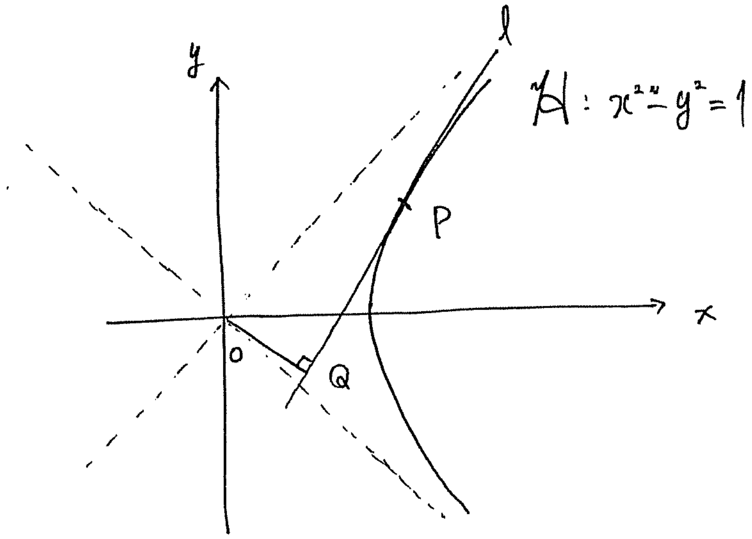
lemniscate
(リボン)

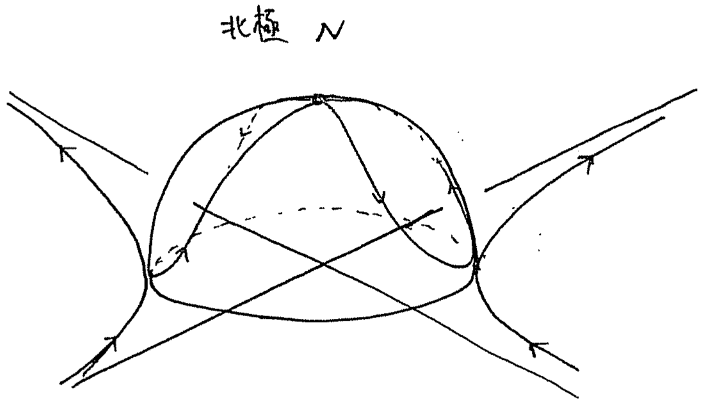
$a=1$として.

$$dS=\frac{1}{2}r^2d\theta$$
$$=\cos 2\theta\, d\theta$$

$$S=4\int_{\theta=0}^{\theta=\frac{\pi}{4}}dS$$
$$=4\int_0^{\frac{\pi}{4}}\cos 2\theta\, d\theta$$
$$=2$$

$\boxed{\text{Task 2-4}}$ 垂足曲線





$P(\alpha,\beta)$, $Q(X,Y)$

$l_P : \alpha x-\beta y=1$

OQ は $\begin{pmatrix}\beta\\ \alpha\end{pmatrix}$ を法線 vector にもつから,

$$\beta X+\alpha Y=0$$

また,

$$X\alpha-Y\beta=1$$

$$X\beta+Y\alpha=0 \iff Y\alpha+X\beta=0$$

$$\alpha=\frac{\begin{vmatrix}1 & -Y\\ 0 & X\end{vmatrix}}{\begin{vmatrix}X & -Y\\ Y & X\end{vmatrix}} \qquad \beta=\frac{\begin{vmatrix}X & 1\\ Y & 0\end{vmatrix}}{\begin{vmatrix}X & -Y\\ Y & X\end{vmatrix}}$$

$$=\frac{X}{X^2+Y^2} \qquad =\frac{-Y}{X^2+Y^2}$$

$P(\alpha,\beta)$ は $\alpha^2-\beta^2=1$ が成立し,

$$X^2-Y^2=(X^2+Y^2)^2$$

$\downarrow$

$X=r\cos\theta$
$Y=r\sin\theta$ とおくと.

$$r^2(\cos^2\theta-\sin^2\theta)=r^4$$

$$\iff \cos 2\theta=r^2$$